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II. Solution by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

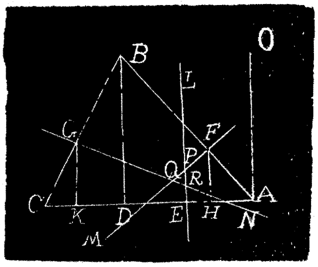
If the random point be taken upon the circumference of a variable circle, concentric with the given circle, the area of the circular "locus" is evidently always one quarter of this variable circle. Again: As the random points are equably distributed over the area of the given circle, the variable circle must always increase in area by equal increments, and, as it varies from zero to the full area of the given circle, its mean area is, of course, half of the given circle. Therefore the mean area of the locus named is one half of one quarter or one eighth of the given circle.

23. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be drawn *perpendicular-sided* to a given plane scalene triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let ABC, be the given scalene triangle; EL, FM, GN the perpendiculars to AC, AB, BC respectively, forming the triangle PQR. Let AC, AO be the axes of co-ordinates. Draw BD, FH, GK perpendicular to AC.



Let $AD=d$, $BC=e$, $BD=h$, $AC=b$, $AB=c$, $BC=a$, $AE=u$, $AH=v$, $AK=w$.

Then $FH=hv/d$, $GK=\dfrac{h(b-w)}{e}$. $x=u$,

$$y=\frac{hv}{d}+\frac{dv}{h}-\frac{dx}{h},\ y=\frac{h(b-w)}{e}+\frac{(b-d)(x-w)}{h},$$

are the equations to LE, FM, and GN respectively.

$$\therefore\ \frac{h^2ev+d^2ev+w(h^2d-d^2e+bde)-h^2db}{bde}=\frac{c^2ev+a^2dw-h^2db}{bde},$$ is the abscissa of Q. Let A_1=area PQR, $\triangle$= area ABC.

$$\therefore\ A_1=\frac{\triangle(c^2ev+a^2dw-h^2db-bdeu)^2}{b^2d^2e^2h^2}=\tfrac{1}{2}\,\frac{(c^2ev+a^2dw-h^2db-bdeu)^2}{bd^2e^2h}$$

The limits of u are 0 and b; of v, 0 and d; of w, d and b.

$$\therefore\ A=\frac{\displaystyle\int_0^b\!\!\int_0^d\!\!\int_d^b A_1\,du\,dv\,dw}{\displaystyle\int_0^b\!\!\int_0^d\!\!\int_d^b du\,dv\,dw}=\frac{1}{bde}\int_0^b\!\!\int_0^d\!\!\int_d^b A_1\,du\,dv\,dw$$

$$=\frac{1}{6a^2b^2d^4e^3h}\int_0^b\!\!\int_0^d\left\{(c^2ev+e^2db-bdeu)^3-(c^2ev-h^2bd+a^2d^2-bdeu)^3\right\}du\,dv$$

$$=\frac{1}{24a^2b^2c^2e^4h}\int_0^b\{(c^2e+e^2b-beu)^4-(e^2b-beu)^4-(c^2e-h^2b+a^2d-beu)^4$$

$$+(a^2d-h^2b-beu)^4 \Big\} du$$

$$=\frac{1}{60a^2b^3c^2h}\Big\{ (a^2+bd)^5-(a^2-be)^5-b^5e^5-b^5d^5 \Big\}$$

$$=\frac{1}{120\triangle a^2b^2c^2}\left\{\left(\frac{a^2+b^2+c^2}{2}\right)^5-\left(\frac{a^2+c^2-b^2}{2}\right)^5-\left(\frac{a^2+b^2-c^2}{2}\right)^5 -\left(\frac{b^2+c^2-a^2}{2}\right)^5\right\}$$

$$=\frac{4\triangle^4}{15a^2b^2c^2}\Big\{(\cot A+\cot B+\cot C)^5-\cot^5 A-\cot^5 B-\cot^5 C\Big\}$$

$$=\frac{a^4+b^4+c^4}{48\triangle}.$$

The last four expressions are the same and show the beautiful relations existing between the terms of the triangle.

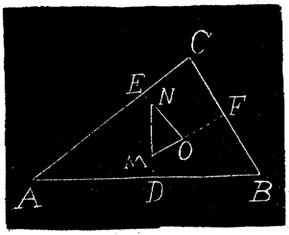
This problem was also solved by *F. P. MATZ.* His solution is published in this issue as solution of problem 24, that problem being identical with 23

24. Proposed by F P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The average area of the triangle formed by three perpendiculars drawn from the sides of the triangle (a, b, c), is $\mathbf{A}=(a^4+b^4+c^4)/48\triangle$.

Solution by the PROPOSER.

Let $BC=a$, $CA=b$, $AB=c$, $BF=x$, $CE=y$, and $AD=z$. The $\triangle MNO$ is similar to the $\triangle ABC$; and it may be *wholly within*, *partly within*, or *wholly without*, the $\triangle ABC$. Put $\angle NAD=\phi$, and $\angle MBD=\theta$; then $AN=(b-y)/\cos(A-\phi)$ $=z\cos\phi \ldots .(\alpha)$, and $ND=z\tan\phi\ldots .(\beta)$. By means of (α), we have from (β), $ND=[(b-y)-z\cos A]/\sin A \ldots .(1)$. Also, $BM=(c-z)/\cos(B-\theta)=x/\cos\theta\ldots .(\gamma)$, and $MD=(c-z)\tan(B-\theta)\ldots .(\delta)$. By means of (γ) we have from (δ), $MD=[x-(c-z)\cos B]/\sin B\ldots .(2)$. Subtracting (2) from (1), we have the expression:



$$MN=\frac{b\sin B+c\sin A\cos B-(x\sin A+y\sin B+z\sin C)}{\sin A\sin B}\ldots .(3).$$

The *first two* terms in the numerator of (3) may be symmetrically written thus: $a\sin B\cos C+b\sin C\cos A+c\sin A\cos B,=\frac{1}{2}(a\sin A+b\sin B+c\sin C)\ldots .(\epsilon)$. Transforming (3) by means of (ϵ), we have

$$MN=\frac{(a-2x)\sin A+(b-2y)\sin B+(c-2z)\sin C}{2\sin A\sin B}\ldots .(4).$$

Representing the numerator of the right-hand member of (4) by $\mathbf{N}$, we have (by symmetry) the expressions:

$NO=\mathbf{N}/2\sin C\sin A$, and $OM=\mathbf{N}/2\sin B\sin C$;